20071010 Princeton colloquium

"Perverse coherent sheaves on a blowup surface"

with Kota Yoshioka

X : complex surface, nonsingular

$x \in X$

$\hat{X} \xrightarrow{p} X$ blowup at x

This is a "surgery" in the category of complex surfaces

- p : isomorphism outside $p^{-1}(x)$
- $p^{-1}(x) \cong \mathbb{CP}^1$
- normal b'dle of $p^{-1}(x)$ = degree -1 line bundle over $\mathbb{CP}^1$

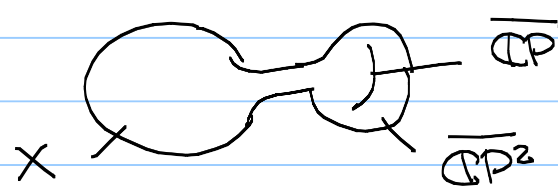
local model

$$\mathbb{C}^2 = \{(x,y)\} \longleftarrow \hat{\mathbb{C}}^2 = \{([z_0 : z_1], x, y) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid z_0 y = z_1 x\}$$

$0 \in \mathbb{C}^2$

If we consider X as a C^∞-4 mfd, then

$$\hat{X} \underset{\text{diffeo.}}{\approx} X \# \overline{\mathbb{CP}^2}$$



Conversely $\underset{\text{curve}}{C} \subset \underset{\text{n.s. cpx surface}}{Y}$

s.t. $C \cong \mathbb{P}^1$, $N_C \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ ((-1)-curve)

$\Rightarrow$ We can collapse C to a point to get a n.s. cpx surface

$$\begin{tikzcd} Y \cong \hat{X} \arrow[r, "p"] & X \\ C \arrow[u, hook] \quad p^{-1}(x) \arrow[u, hook] & x \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

We can further collapse a curve with the same property. Then get a surface without (-1)-curve.
minimal model

▷ Key tool of the classification of cpt cpx surface by Kodaira

The relation between X & $\hat{X}$ is very simple.

e.g.

$$H_2(\hat{X}, \mathbb{R}) = H_2(X, \mathbb{R}) \oplus \mathbb{R}[C]$$

orthogonal w.r.t. intersection pairing

$$K_{\hat{X}} = p^* K_X + C \quad \text{(canonical b'dle)}$$

Next we want to understand relation between holomorphic vector bundles on X and those of $\hat{X}$.

$E \to X$ vector bundle on X

$\Rightarrow p^*E$: vector bundle on X

Further we can "twist" a vector bundle F on $\hat{X}$ along C ("elementary transformations")

(Prototype: divisor $D \longleftrightarrow$ line b'dle $\mathcal{O}(-D)$

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

F : vector bundle on $\hat{X}$

$\mathcal{S} \subset F|_C$ subbundle

$\rightsquigarrow$ F' : new vector bundle on $\hat{X}$

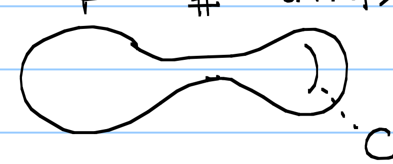
$$\mathcal{O}(F') = \{ s \in \mathcal{O}(F) \mid s|_C \in \mathcal{S} \}$$

$$0 \to \mathcal{O}(F') \to \mathcal{O}(F) \to \mathcal{O}_C(\mathcal{S}) \to 0$$

⊙ F & F' are isomorphic outside C, but twisted along C

F # ext. of $\mathcal{S}$

topologically



$C \cong \mathbb{P}^1$ <u>Grothendieck</u> :

classification of vector b'dles on C

$$\mathcal{E} \cong \mathcal{O}_C(d_1) \oplus \cdots \oplus \mathcal{O}_C(d_r)$$

$$d_i \in \mathbb{Z}$$

Thus vector b'dles on $\mathbb{P}^1$ are easy to understand.

Considering the restriction to C, it is not difficult to show :

Any vector b'dle F on $\hat{X}$ is obtained from p^*E by successive applications of elementary transformations.

Thus we can understand a vector b'dle on $\hat{X}$ to some extent.

<u>Rem.</u> $\mathcal{O}_C(\mathcal{E})$ appears as a quotient, not as a sub.

$\times$ $0 \to \mathcal{O}_C(\mathcal{E}) \xrightarrow{i} \mathcal{O}(F') \longrightarrow \mathcal{O}(F) \to 0$

$\quad\quad \underset{s}{\cup}$

any section $s \Rightarrow i(s)$ vanish outside C locally $\cong \mathcal{O}_X^{\oplus r}$

$\Rightarrow$ vanish also on C as F': v.b.

Later we allow such an object as a perverse coherent sheaf. ← THIS CANNOT BE A VECT. BUNDLE (NOR A TORSION-FREE SHEAF).

From the above consideration, it is clear that the elementary transformation is "local". (dep. on a n.b.d. of C, $F|C$, δ)
independent of a surface X.

$\rightsquigarrow$ $\underbrace{\text{"Blowup formula"}}_{\text{comparing invariants of } X \text{ \& } \hat{X}}$ is universal indep. of X.

Thus the blowup formula is built in the theory.

Next we want to study relations between moduli spaces of vector b'dles on X & $\hat{X}$.

$X, \hat{X}$: projective surfaces
H : ample line b'dle on X hereafter

$M^{\circ}_{X,H}(r, c_1, c_2)$ = moduli space of H-stable vect. bdle E on X
$\operatorname{rk} E = r$, $c_1(E) = c_1$, $c_2(E) = c_2$.
$\cap$
$M_{X,H}(r, c_1, c_2)$ = Gieseker-Maruyama compactification
= moduli of H-ss. sheaves

H-(semi)stability ---- technical notion, used to define the moduli space as a scheme via a GIT quotient

$\Leftrightarrow$ $\underset{\text{subsheaf}}{F \subset E}$ $\quad \dfrac{\chi(F \otimes H^k)}{\operatorname{rk} F} \underset{(=)}{<} \dfrac{\chi(E \otimes H^k)}{\operatorname{rk} E}$ for $k \gg 0$

p^*H on $\hat{X}$ is not ample, but is a limit of $\mathbb{Q}$-ample line b'dles $\lim_{\varepsilon \to +0} (p^*H - \varepsilon C)$

If $0 < \varepsilon \ll 1$, $M_{\hat{X}, H-\varepsilon C}$ is independent of ε. We denote it by $M_{\hat{X}, H}$.

$$M_{X,H}(r, c_1, c_2) \overset{p^*}{\dashrightarrow} M_{\hat{X},H}(r, c_1, c_2) \quad \text{birational}$$

The complement can be obtained from $M_{X,H}(r, c_1', c_2')$ by $\underbrace{\text{elem. transf.}}_{\text{succ.}}$

<u>Th.</u> (Yoshioka)

$$\frac{\sum \chi(M_{\hat{X},H}(r, c_1+kC, c_2))\, q^{c_2}}{\sum \chi(M_{X,H}(r, c_1, c_2))\, q^{c_2}}$$

is independent of X
is given by
a theta function
($\mathbb{Z}^r$-lattice)

⊙ Euler # can be computed by cutting the moduli spaces.

But more "global" understanding of relations between moduli spaces seems necessary to compute blowup formula for finer invariants :

e.g. • Donaldson invariants
= intersection product of cycles on moduli spaces M_H

$$\int_{M_H} \exp(c_2(\mathcal{E})/\alpha) \qquad \alpha \in H_*(X)$$

$\mathcal{E} \to X \times M_H$ universal sheaf

• equivariant Donaldson inv.
($\subset_?$ Donaldson inv. of families of 4-mfds)

• K-theoretic Donaldson invariants
$\mathcal{L}$: determinant line bundle on M_H

$$\chi(M_H, \mathcal{L}^{\otimes n}) = ?$$

Q. Why do we care blowup formula?

No theoretical explanation:
only from experiences

— Fintushel–Stern gave the blowup formula
for Donaldson invariants:

D. inv. of $\hat{X} = X \# \overline{\mathbb{CP}^2}$
can be written in terms of D. inv. of X
& Coeff.'s are written in terms of elliptic functions.

$\boxed{r=2}$

$$\sum_{c_2} \int_{M_{\hat{X},H}(r,c_1,c_2)} \exp\left(c_2(\mathcal{E})/(\alpha + C)\right) t^{c_2} = \int_{M_{X,H}(r,c_1,c_2)} \exp\left(c_2(\mathcal{E})/\alpha\right) B\left(\frac{c_2(\mathcal{E})}{[x]}\right) t^{c_2}$$

t: variable
$\alpha \in H_*(X)$

$B(u)$: written in terms of elliptic integral ass. with the elliptic curve:
$y^2 = (z^2 + u + 1)(z^2 + u - 1)$

The appearance of an elliptic curve was very surprising, as we cannot see it in the definition of invariants. But it was one of keys to understand the structure of Donaldson invariants
(Seiberg–Witten curves)

"mirror"

— equivariant Donaldson inv.

blowup formula $\Rightarrow$ functional equation

for equiv. Donald. inv of $\mathbb{R}^4$

(Nekrasov partition function)

A new approach:

we realise $M_{X,H}$ as a moduli space of "objects" on $\hat{X}$.

More precisely we also have intermediate moduli spaces of objects on $\hat{X}$ connecting $M_{X,H}$ & $M_{\hat{X},H}$

$$\begin{tikzcd}[column sep=small]
 & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & \\
M_{\hat{X},H} & & M_1 & & M_2 & & \cdots & & M_N = M_{X,H}
\end{tikzcd}$$

$$\begin{array}{c} \| \\ M_0 \end{array}$$

& $M_i \sim M_{i+1}$ are related by a simpler birational transformation which can be understood.

Idea: Bridgeland perverse coherent sheaf

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] & \\
C \subset X^3 \arrow[dr] & & X^+ \arrow[dl] \\
 & Y &
\end{tikzcd}$$

$N = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$

flop

one of the fund. operations in 3D birational morphisms

X & X' are isomorphic outside subvar. of curves.

$D(\mathrm{Coh}\, X^+) \cong D(\mathrm{Coh}\, X)$

realise X^+ = moduli space of perverse coherent ideal sheaves on X

$x \in X \setminus C$ $\quad I_x$: perverse coh.

but $x \in C$ $\quad I_x$: not

$x^+ \in C^+ \rightsquigarrow$ some funny cpx on X

The Fourier transform w.r.t. universal sheaf gives the derived equiv.

$p : \hat{X} \longrightarrow X$ $\quad H$: as before

F : a coherent sheaf on $\hat{X}$ is H – stable & perverse

(this is a combination of two conditions)

(1) $\mathrm{Hom}(F, \mathcal{O}_C(-1)) = 0$
(2) $p_* F$ is H-stable torsion-free sheaf on X

$M_m(r, c_1, c_2) = \{ F \mid F \otimes \mathcal{O}(-mC)$ is H-stable perverse coherent $\}$

$c_i \in H_*(X)$

Pf (1) $m \gg 0$ (compared with r, c_1, c_2)

$$M_m(r, c_1, c_2) = M_{\hat{X}, H}(r, c_1, c_2)$$

(2) (Assume $(c_1, C) = 0$.)

Then $M_0(r, c_1, c_2) = M_{X, H}(r, c_1, c_2)$